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Partial order in the self-dual Ashkin–Teller model

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Abstract. Corner transfer matrices are used to calculate the order parameters of the self-dual Ashkin-Teller model on the square lattice. In the non-critical regime, it is found that the model is partially ordered, i.e. the magnetisations $\langle s_1\rangle$ and $\langle t_1\rangle$ vanish while the polarisation $\langle s_1 t_1\rangle$ is non-zero. The polarisation, which is given simply in terms of elliptic functions, exhibits an essential singularity at criticality.

1. Introduction

The complete phase diagram of the isotropic square lattice Ashkin-Teller model has been deduced by the application of a variety of techniques. Arguments using duality (Fan and Wu 1970), continuity (Wu and Lin 1974), the study of limiting cases (Knops 1975), series analysis (Ditzian *et al* 1980), as well as correlation inequalities (Pfister 1982), have established the general features of the phase diagram and the various types of ordering. A succinct summary of these results for the isotropic Ashkin-Teller model can be found in Baxter (1982) and a phase diagram is shown in figure 1.

The isotropic Ashkin-Teller model is known to be exactly solvable along its self-dual line (Fan 1972a). Recently, Pearce (1987) has obtained a two-dimensional exact solution manifold for the anisotropic model which, when restricted to isotropic interactions, coincides with the self-dual line. Most importantly, however, this more general solution allows commuting transfer matrix methods to be applied to the Ashkin-Teller model. The key to the study of the commuting row-to-row transfer matrices is the inversion identity as discussed in Pearce (1987).

In this paper we apply commuting corner transfer matrix techniques to the anisotropic Ashkin-Teller model to obtain exact expressions for the order parameters on the exact solution manifold. In § 2 we define the model, recall its parametrisation and verify the star-triangle or Yang-Baxter relations. Appropriate corner transfer matrices are set up and diagonalised in § 3. Finally, the order parameters are obtained in § 4 and the limiting behaviour of the polarisation is examined.

In a forthcoming paper (Pearce and Seaton 1989) we obtain the order parameters for a more general off-critical integrable extension of the Ashkin-Teller model and discuss these results in the light of recent work on conformal invariance.

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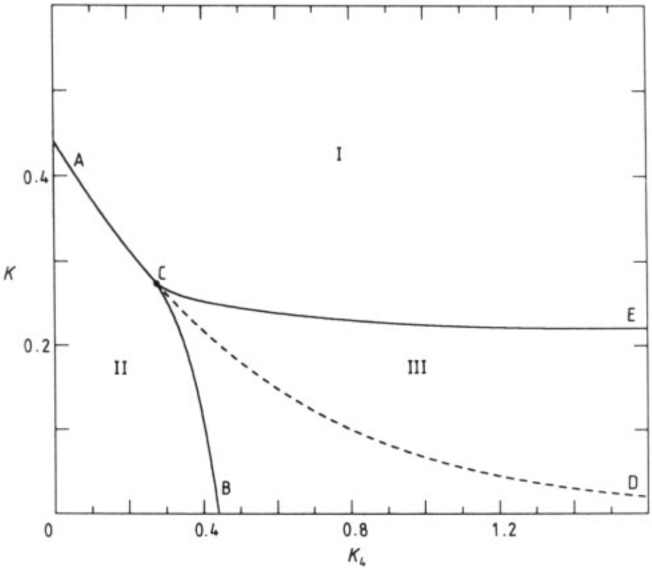

Figure 1. The phase diagram of the ferromagnetic isotropic Ashkin–Teller model in the thermodynamic space spanned by the two-spin interaction K and the four-spin interaction K_4. The regions marked I, II and III are ferromagnetically ordered, disordered and partially ordered, respectively. The line ACD is the self-dual line given by $\sinh 2K = \exp(-2K_4)$. The segment CD, shown broken, is the partially ordered line studied here. AC is a critical line with continuously varying exponents. The lines CB and CE, shown schematically, are lines of Ising-like critical behaviour.

2. The model and star–triangle relations

The square lattice Ashkin–Teller model is an interaction-round-a-face or IRF model (Baxter 1982). Each site i of the lattice is occupied by a four-state spin $\sigma_i = 1, 2, 3, 4$. Because of the symmetries of the model, it is convenient (Fan 1972b) to represent each four-state spin as a compound spin $\sigma_i = (s_i, t_i)$ where s_i, $t_i = \pm 1$ are Ising spins. We will use the correspondence $1 = (+,+)$, $2 = (-,-)$, $3 = (-,+)$, $4 = (+,-)$. The Boltzmann weight of a square face (i, j, k, l), as shown in figure 2, can then be written in terms of compound spins as

$$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = W_1(\sigma_i, \sigma_k)\, W_2(\sigma_j, \sigma_l) \tag{2.1}$$

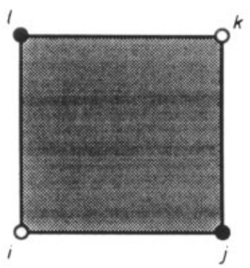

Figure 2. A face of the square lattice surrounded by the lattice sites i, j, k, l. The Boltzmann weight of the face factors over the two sublattices shown by full and open circles.

where

$$W_1(\sigma_i, \sigma_j) = \rho_1 \exp[J(s_i s_j + t_i t_j) + J_4 s_i s_j t_i t_j] r_{\sigma_i} r_{\sigma_j} \tag{2.2a}$$

$$W_2(\sigma_i, \sigma_j) = \rho_2 \exp[K(s_i s_j + t_i t_j) + K_4 s_i s_j t_i t_j] / r_{\sigma_i} r_{\sigma_j}. \tag{2.2b}$$

Here J, K are two-spin interactions, J_4 and K_4 are four-spin interactions, the r_{σ_i} are arbitrary gauge factors and ρ_1, ρ_2 are normalisation factors.

The square lattice consists of two interpenetrating sublattices which we label 0 and 1. Because the interactions (2.2) act only across the diagonals of the square faces, the Ashkin-Teller model, as defined, is actually a superposition of two independent square lattice models with nearest-neighbour interactions, one on each sublattice. The Ashkin-Teller interactions are thus invariant under four independent partial spin-reversal symmetries given by $s_i \to -s_i$ or $t_i \to -t_i$ for all sites i on either sublattice 0 or sublattice 1. For ferromagnetic interactions, i.e. when J, K, J_4, $K_4 > 0$, there are therefore sixteen ground-state configurations obtained from the uniform configuration $\sigma_i = (+, +)$ by application of the four partial spin-reversal symmetries to the two sublattices.

As given in Pearce (1987), the two-dimensional exact solution manifold of the anisotropic Ashkin-Teller model is determined by the two constraints

$$\frac{\sinh 2J_4}{\sinh 2J} = \frac{\sinh 2K_4}{\sinh 2K} = \Delta/2 \tag{2.3}$$

$$[\exp(4J_4) - 1][\exp(4K_4) - 1] = \Delta^2 \tag{2.4}$$

among the four thermodynamic parameters J, K, J_4, K_4. Alternatively, the exact solution manifold can be parametrised in terms of a spectral parameter u and a crossing parameter λ. This parametrisation is defined by the relations

$$\Delta = \begin{cases} 2\cos\lambda & 0 \leqslant \Delta \leqslant 2 \\ 2\cosh\lambda & \Delta > 2 \end{cases} \tag{2.5}$$

and

$$s(u) = \begin{cases} \sin u & 0 \leqslant \Delta \leqslant 2 \\ \sinh u & \Delta > 2 \end{cases} \tag{2.6}$$

with

$$s = s(u)/s(\lambda) = \tanh 2K \qquad s_- = s(\lambda - u)/s(\lambda) = \tanh 2J. \tag{2.7}$$

Then, showing the dependence on the gauge factors r and the spectral parameter u explicitly, the weights are parametrised by

$$W_1(\sigma_i, \sigma_j | r, u) = r_{\sigma_i} r_{\sigma_j} [(1+s) + s_-(s_i s_j + t_i t_j) + (1-s) s_i s_j t_i t_j]/2\sqrt{2} \tag{2.8}$$

$$W_2(\sigma_i, \sigma_j | r, u) = W_1(\sigma_i, \sigma_j | r^{-1}, \lambda - u) \tag{2.9}$$

where we have chosen the normalisation factors:

$$\rho_1 = (s/2)^{1/2}(1 - s_-^2)^{1/4} \qquad \rho_2 = (s_-/2)^{1/2}(1 - s^2)^{1/4}. \tag{2.10}$$

These weights possess the rotation or crossing symmetry:

$$W(\sigma_j, \sigma_k, \sigma_l, \sigma_i | r, u) = W(\sigma_i, \sigma_j, \sigma_k, \sigma_l | r^{-1}, \lambda - u). \tag{2.11}$$

They also satisfy inversion relations as stated in Pearce (1987).

The star-triangle or Yang-Baxter relations (Baxter 1982) for the Ashkin-Teller model take the form

$$\sum_{\tau} W_1(\sigma, \tau|u)\, W_1(\sigma', \tau|u')\, W_1(\sigma'', \tau|u'')$$
$$= A\, W_1(\sigma, \sigma'|\lambda - u'')\, W_1(\sigma', \sigma''|\lambda - u)\, W_1(\sigma'', \sigma|\lambda - u') \tag{2.12}$$

where A is a constant to be determined and

$$u + u' + u'' = \lambda. \tag{2.13}$$

These equations must hold for all values of the spins σ, σ', σ''. There are 4^3 choices of σ, σ', σ'' but the symmetries of the weights and of (2.13) reduce the number of independent cases to be considered. Up to permutations of unprimed, primed and double primed variables, there are four distinct cases:

$$\begin{aligned} &\sigma = \sigma' = \sigma'' = 1 && \sigma = \sigma' = 1,\ \sigma'' = 3 \\ &\sigma = \sigma' = 1,\ \sigma'' = 2 && \sigma = 1,\ \sigma' = 2,\ \sigma'' = 3. \end{aligned} \tag{2.14a}$$

Substituting these four cases into (2.12) and using the obvious notation

$$s' = \frac{s(u')}{s(\lambda)} \qquad s'_- = \frac{s(\lambda - u')}{s(\lambda)} \qquad s'' = \frac{s(u'')}{s(\lambda)} \qquad s''_- = \frac{s(\lambda - u'')}{s(\lambda)} \tag{2.14b}$$

gives

$$1 + s_-s'_- + s'_-s''_- + s''_-s_- + ss's'' = A(1+s)(1+s')(1+s'')/2 \tag{2.15a}$$

$$1 + s_-s'_- - s'_-s''_- - s''_-s_- + ss's'' = A(1+s'')(1-s')(1-s)/2 \tag{2.15b}$$

$$s'' + s_-s'_-s'' + ss' = As_-s'_-(1+s'')/2 \tag{2.15c}$$

$$s'' - s_-s'_-s'' + ss' = As_-s'_-(1-s'')/2. \tag{2.15d}$$

These equations are satisfied if $A = 2$. With this choice, the last two, and the sum of the first two equations all simplify to

$$s_-s'_- = s'' + ss' \tag{2.16}$$

or equivalently

$$s(\lambda - u)s(u + u'') = s(\lambda)s(u'') + s(u)s(\lambda - u - u'') \tag{2.17}$$

which can be proved using Liouville's theorem or trigonometric ($0 \leqslant \Delta \leqslant 2$) and hyperbolic ($\Delta > 2$) identities. The difference of (2.15a) and (2.15b) is

$$s'_-s''_- + s_-s''_- = s + s's'' + s' + ss'' \tag{2.18}$$

which follows from (2.17) and (2.13).

If $u = \lambda/2$ then $s = s_-$ and from (2.7) and (2.3) the interactions are isotropic. In this case (2.3) and (2.4) give

$$\sinh 2K = \exp(-2K_4) \tag{2.19}$$

which is the self-duality condition for the isotropic model (Fan 1972a). If $K_4 < K$, which corresponds to $\Delta < 2$, this line is critical (Baxter 1982) and $\langle s_1 \rangle = \langle t_1 \rangle = \langle s_1 t_1 \rangle = 0$. Otherwise, if $\Delta > 2$, this self-duality line lies entirely within the partially ordered region of the phase diagram where $\langle s_1 \rangle = \langle t_1 \rangle = 0$ and $\langle s_1 t_1 \rangle \neq 0$. In calculating the order parameters, we thus confine our attention to the case $\Delta > 2$ on the exact solution manifold of the anisotropic Ashkin-Teller model. In this case the parametrisation is hyperbolic.

3. Corner transfer matrix calculations

Let us define corner transfer matrices (CTM) A, B, C, D in the usual way (Baxter 1981) so that A corresponds to the lower right quadrant of the lattice. The (σ, σ') element of A is

$$A(r, u)_{\sigma,\sigma'} = \delta(\sigma_1, \sigma'_1) \sum_{\sigma} \prod_{(i,j,k,l)} W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \tag{3.1}$$

where the product is over all faces of the quadrant, the sum is over all interior spins and $\sigma = (\sigma_1, \ldots, \sigma_m)$ and $\sigma' = (\sigma'_1, \ldots, \sigma'_m)$ are the edge spins as shown in figure 3. Similarly, we define corner transfer matrices B, C, D corresponding, respectively, to the upper right, upper left and lower left quadrants of the lattice. The two magnetisations of the Ashkin-Teller model can then be written as

$$\langle s_1 \rangle = \frac{\mathrm{Tr}\, SABCD}{\mathrm{Tr}\, ABCD} \qquad \langle t_1 \rangle = \frac{\mathrm{Tr}\, TABCD}{\mathrm{Tr}\, ABCD} \tag{3.2}$$

where the elements of the matrices S and T are

$$S_{\sigma,\sigma'} = s_1 \prod_{i=1}^{m} \delta(\sigma_i, \sigma'_i) \qquad T_{\sigma,\sigma'} = t_1 \prod_{i=1}^{m} \delta(\sigma_i, \sigma'_i) \tag{3.3}$$

and the polarisation is

$$\langle s_1 t_1 \rangle = \frac{\mathrm{Tr}\, STABCD}{\mathrm{Tr}\, ABCD}. \tag{3.4}$$

From the crossing symmetry, it follows that

$$C(r, u) = A(r, u) \qquad B(r, u) = D(r, u) = gA(r^{-1}, \lambda - u) \tag{3.5}$$

where g is a gauge factor arising from boundary spins.

In the limit of m large, the corner transfer matrices can be diagonalised (Baxter 1982). In particular, the diagonal entries or eigenvalues of A take the simple form

$$A(r, u)_{\sigma,\sigma} = m_\sigma \exp(-\alpha_\sigma u) \tag{3.6}$$

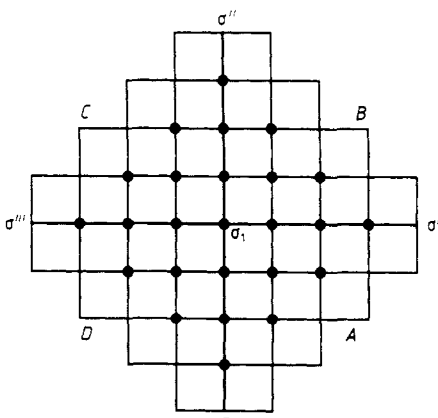

Figure 3. The corner transfer matrices A, B, C, D corresponding to the four quadrants of the square lattice. The central spin is σ_1. The spins on internal sites, shown with full circles, are summed over while the spins on the perimeter are fixed to the boundary values b, c, respectively, on the two sublattices.

where the values of the constants m_σ and α_σ can be determined from special limiting cases. Noting that

$$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l|r, 0) = W_1(\sigma_i, \sigma_k|r, 0)\, W_1(\sigma_j, \sigma_l|1/r, \lambda) = \frac{r_{\sigma_i} r_{\sigma_k}}{r_{\sigma_j} r_{\sigma_l}} \delta_{\sigma_i \sigma_k} \tag{3.7}$$

it follows, from (3.1), that

$$A(r, 0)_{\sigma,\sigma'} = \frac{1}{hr_{\sigma_1}} \prod_{i=1}^{m} \delta(\sigma_i, \sigma_i') \tag{3.8}$$

where h is a gauge factor arising from boundary sites. We therefore identify

$$m_\sigma = 1/hr_{\sigma_1}. \tag{3.9}$$

To determine the remaining constant α_σ, we first define

$$x = \mathrm{e}^{-\lambda} \qquad w = \mathrm{e}^{-u}. \tag{3.10}$$

In the limit $\lambda \to \infty$ or $x \to 0$ we then find

$$r_{\sigma_i} r_{\sigma_j} W_1(\sigma_i, \sigma_j|1/r, \lambda - u) \to \begin{cases} 1 & s_i s_j t_i t_j = 1 \\ w & s_i s_j t_i t_j = -1 \end{cases} \tag{3.11}$$

and

$$W_1(\sigma_i, \sigma_j|r, u) \to W_{\sigma_i \sigma_j} r_{\sigma_i} r_{\sigma_j} \tag{3.12}$$

where $W_{\sigma_i \sigma_j}$ are the elements of the matrix

$$W = \frac{1}{2} \begin{pmatrix} 1+w & 1-w & 0 & 0 \\ 1-w & 1+w & 0 & 0 \\ 0 & 0 & 1+w & 1-w \\ 0 & 0 & 1-w & 1+w \end{pmatrix}. \tag{3.13}$$

This matrix is easily diagonalised as $W = P^{-1}DP$ where

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \tag{3.14}$$

and the orthogonal matrix P is given by

$$P = P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{3.15}$$

Hence in the limit $\lambda \to \infty$, we find

$$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = w^{n(\sigma_l, \sigma_i, \sigma_j)} \frac{r_{\sigma_i}^2}{r_{\sigma_j} r_{\sigma_l}} \delta(\sigma_i, \sigma_k) \tag{3.16}$$

where

$$n(\sigma_l, \sigma_i, \sigma_j) = 1 - (t_i + s_j s_l t_j t_l)/2. \tag{3.17}$$

We conclude that

$$\alpha_\sigma = \sum_{j=1}^{m} jn(\sigma_j, \sigma_{j+1}, \sigma_{j+2}) \tag{3.18}$$

where the values of σ_{m+1}, σ_{m+2} are fixed by the boundary conditions. The eigenvalues of A are therefore given by

$$A_{\sigma,\sigma} = \frac{1}{hr_{\sigma_1}} \prod_{j=1}^{m} w^{jn(\sigma_j,\sigma_{j+1},\sigma_{j+2})}. \tag{3.19}$$

Let us define the one-dimensional configuration sums:

$$X_m^{\sigma_1\sigma_{m+1}\sigma_{m+2}} = \sum_{\sigma_2\ldots\sigma_m} q^{\alpha_\sigma} \tag{3.20}$$

where $q = x^2$. Then from equations (3.2)–(3.5) it follows that

$$\langle s_1 \rangle = \lim_{m\to\infty} \frac{\sum_{\sigma_1} s_1 X_m^{\sigma_1\sigma_{m+1}\sigma_{m+2}}}{\sum_{\sigma_1} X_m^{\sigma_1\sigma_{m+1}\sigma_{m+2}}} \tag{3.21}$$

$$\langle t_1 \rangle = \lim_{m\to\infty} \frac{\sum_{\sigma_1} t_1 X_m^{\sigma_1\sigma_{m+1}\sigma_{m+2}}}{\sum_{\sigma_1} X_m^{\sigma_1\sigma_{m+1}\sigma_{m+2}}} \tag{3.22}$$

$$\langle s_1 t_1 \rangle = \lim_{m\to\infty} \frac{\sum_{\sigma_1} s_1 t_1 X_m^{\sigma_1\sigma_{m+1}\sigma_{m+2}}}{\sum_{\sigma_1} X_m^{\sigma_1\sigma_{m+1}\sigma_{m+2}}}. \tag{3.23}$$

The one-dimensional configuration sums X_m^{abc} satisfy the linear recursion relations

$$X_m^{abc} = \sum_{\sigma_m=1}^{4} q^{mn(\sigma_m,b,c)} X_{m-1}^{a\sigma_m b} \tag{3.24}$$

and the initial conditions

$$X_1^{abc} = q^{n(a,b,c)} \tag{3.25}$$

where a, b and c denote general values of the four-state spins. These equations also imply the simple alternative initial condition:

$$X_0^{abc} = \delta(a, b) \tag{3.26}$$

where δ is the Kronecker delta symbol. From the symmetries of $n(a, b, c)$ we see that

$$\begin{aligned} &X_m^{1bc} = X_m^{2bc} && X_m^{3bc} = X_m^{4bc} \\ &X_m^{a1c} = q^{-m} X_m^{a2c} && X_m^{a3c} = q^{-m} X_m^{a4c} \\ &X_m^{ab1} = X_m^{ab2} && X_m^{ab3} = X_m^{ab4} \end{aligned} \tag{3.27}$$

so the recursion relations reduce to

$$\begin{aligned} X_m^{a11} &= (1+q^m)(X_{m-1}^{a11} + q^m X_{m-1}^{a31}) \\ X_m^{a13} &= (1+q^m)(X_{m-1}^{a31} + q^m X_{m-1}^{a11}) \\ X_m^{a31} &= (1+q^m)(X_{m-1}^{a13} + q^m X_{m-1}^{a33}) \\ X_m^{a33} &= (1+q^m)(X_{m-1}^{a33} + q^m X_{m-1}^{a13}). \end{aligned} \tag{3.28}$$

These relations and the initial conditions are unaltered if 1 is interchanged with 3, so it follows that

$$\begin{aligned} X_m^{311} &= X_m^{133} & X_m^{313} &= X_m^{131} \\ X_m^{333} &= X_m^{111} & X_m^{331} &= X_m^{113}. \end{aligned} \tag{3.29}$$

Similarly, because $X_1^{111} = X_1^{131}$ and $X_1^{113} = X_1^{133}$, it can be shown that

$$\begin{aligned} X_{2m+1}^{111} &= X_{2m+1}^{131} & X_{2m+1}^{133} &= X_{2m+1}^{113} \\ X_{2m}^{111} &= X_{2m}^{113} & X_{2m}^{133} &= X_{2m}^{131} \end{aligned} \tag{3.30}$$

leaving only 2 of the original 64 one-dimensional configuration sums X_m^{abc} to be determined.

Next, let us define the polynomial F_n by

$$F_n(x) = \prod_{j=1}^{n} (1+x^j) \tag{3.31}$$

and let $[\ldots]$ denote the integer part function. Then the forms

$$\begin{aligned} X_m^{111} &= \frac{1}{1+q} F_m(q) F_{[m/2]}(q^2)\, Y_{[(m+1)/2]}(q) \\ X_m^{133} &= \frac{q}{1+q} F_m(q) F_{[m/2]}(q^2) Z_{[(m+1)/2]}(q) \end{aligned} \tag{3.32}$$

in conjunction with (3.30) satisfy the recursion relations (3.28) provided the functions $Y_m(q)$ and $Z_m(q)$ in turn satisfy the recursion relations

$$\begin{pmatrix} Y_m \\ qZ_m \end{pmatrix} = \begin{pmatrix} 1 & q^{2m-1} \\ q^{2m-1} & 1 \end{pmatrix} \begin{pmatrix} Y_{m-1} \\ qZ_{m-1} \end{pmatrix} \tag{3.33}$$

subject to the initial conditions $Y_1 = Z_1 = 1$.

The polynomials $Y_m(q)$ and $Z_m(q)$ can easily be obtained by diagonalisation and iteration. This procedure yields the solutions

$$\begin{aligned} Y_m(q) + qZ_m(q) &= \prod_{n=1}^{m} (1+q^{2n-1}) \\ Y_m(q) - qZ_m(q) &= \prod_{n=1}^{m} (1-q^{2n-1}) \end{aligned} \tag{3.34}$$

or

$$\begin{aligned} Y_m(q) &= \frac{1}{2}\left[\prod_{n=1}^{m} (1+q^{2n-1}) + \prod_{n=1}^{m} (1-q^{2n-1})\right] \\ qZ_m(q) &= \frac{1}{2}\left[\prod_{n=1}^{m} (1+q^{2n-1}) - \prod_{n=1}^{m} (1-q^{2n-1})\right]. \end{aligned} \tag{3.35}$$

In the thermodynamic limit $m \to \infty$ these functions are simply related to Virasoro characters as will be discussed in Pearce and Seaton (1989).

4. The order parameters

It is readily seen that the two magnetisations:

$$\langle s_1\rangle = \lim_{m\to\infty} \frac{X_m^{1bc} - X_m^{2bc} - X_m^{3bc} + X_m^{4bc}}{X_m^{1bc} + X_m^{2bc} + X_m^{3bc} + X_m^{4bc}} \tag{4.1a}$$

$$\langle t_1\rangle = \lim_{m\to\infty} \frac{X_m^{1bc} - X_m^{2bc} + X_m^{3bc} - X_m^{4bc}}{X_m^{1bc} + X_m^{2bc} + X_m^{3bc} + X_m^{4bc}} \tag{4.1b}$$

both vanish as a consequence of the symmetries (3.27). On the other hand, using (3.27), the polarisation is given by

$$\langle s_1 t_1\rangle = \lim_{m\to\infty} \frac{X_m^{1bc} - X_m^{3bc}}{X_m^{1bc} + X_m^{3bc}}. \tag{4.2}$$

Clearly the one-dimensional configuration sums X_m^{abc} depend on the boundary conditions b, c and on the parity of m. To remove any ambiguity, we apply boundary conditions so that b is on the sublattice labelled 0. The polarisation on the ε sublattice, where $\varepsilon = 0$ or 1, with boundary conditions b, c is then

$$\langle s_1 t_1\rangle_\varepsilon^{bc} = \lim_{\substack{m\to\infty \\ m=\varepsilon \bmod 2}} \frac{X_m^{1bc} - X_m^{3bc}}{X_m^{1bc} + X_m^{3bc}}.$$

$$= \mu_\varepsilon \lim_{m\to\infty} \frac{Y_{[(m+1)/2]} - qZ_{[(m+1)/2]}}{Y_{[(m+1)/2]} + qZ_{[(m+1)/2]}} = \mu_\varepsilon \prod_{n=1}^{\infty} \frac{1-q^{2n-1}}{1+q^{2n-1}} \tag{4.3}$$

where the known symmetries have been used and the sign factor $\mu_\varepsilon = \pm 1$ is given by

$$\mu_\varepsilon = st \tag{4.4}$$

where s, t are the two Ising spin components of b if $\varepsilon = 0$ or c if $\varepsilon = 1$. The sums involved are particular cases of the elliptic function:

$$E(y,z) = \sum_{n=-\infty}^{\infty} (-1)^n y^n z^{n(n-1)/2} = \prod_{n=1}^{\infty} (1-z^{n-1}y)(1-z^n/y)(1-z^n). \tag{4.5}$$

So finally we obtain

$$\langle s_1 t_1\rangle_\varepsilon^{bc} = \mu_\varepsilon \left(\frac{E(x^2, x^4)}{E(-x^2, x^4)}\right)^{1/2} = \mu_\varepsilon \prod_{n=1}^{\infty} \frac{1-x^{4n-2}}{1+x^{4n-2}} \tag{4.6}$$

where we have eliminated q in favour of $x = e^{-\lambda}$ using $q = x^2$. Clearly the polarisation, unlike the magnetisations, is in general non-zero so partial order is established at least in the regime $\Delta > 2$ of the exact solution manifold of the anisotropic Ashkin-Teller model.

To examine the limiting behaviour of the polarisation as criticality is approached, i.e. as $\lambda \to 0$ or $x \to 1$, we need to perform a conjugate modulus transformation of the form

$$E(\exp(2ui-\varepsilon), \exp(-2\varepsilon)) = (\pi/\varepsilon)^{1/2} \exp[-(u-\pi/2)^2/\varepsilon] E(-\exp(-2\pi u/\varepsilon), \exp(-2\pi^2/\varepsilon)). \tag{4.7}$$

This gives

$$\langle s_1 t_1\rangle_\varepsilon^{bc} = \mu_\varepsilon \exp(-\pi^2/16\lambda) \left(\frac{E(-1, \exp(-\pi^2/\lambda))}{E(-\exp(-\pi^2/2\lambda), \exp(-\pi^2/\lambda))}\right)^{1/2}. \tag{4.8}$$

Clearly the polarisation vanishes as $\lambda \to 0$. Moreover, since the variable which measures deviation-from-criticality is λ^2, we see that the point $\lambda = 0$ is an essential singularity.

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